A Little Bit of Measure Theory

Lecture 6: Continuum Economies

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Core and competitive equilibria

Definition

An allocation x is in the core if no coalition of agents can do better than x for all its members by just using its own resources.

Question

Every competitive equilibrium is in the core. Is everything in the core a competitive equilibrium?

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Ans: Only for a 'large' economy!

The setup

- A pure exchange economy.
- The set of commodity bundles is the positive orthant Ω in $\mathbb{R}^n$ for some n.
- The set of consumer is $T = [0, 1]$ with the Lebesgue measure.
- An assignment is an integrable map from T to Ω.
- There is a fixed assignment $\mathbf{i}(t)$ such that $\int_T \mathbf{i} > 0$.
- An allocation is an assignment $\mathbf{i}$ for which $\int_T \mathbf{x} = \int_T \mathbf{i}$.

Assumptions

- Desirability $x \geqq y$ implies $x \succ_t y$.
- Continuity For each $y \in \Omega$ the sets $\{x \colon x \succ_t y\}$ and $\{x \colon y \succ_t x\}$ are open.
- Measurability If $\mathbf{x}$ and $\mathbf{y}$ are assignments, then the set $\{t \colon \mathbf{x}(t) \succ_t \mathbf{y}(t)\}$ is Lebesgue measurable in T.

The core

Definition

A *coalition* of traders is a Lebesgue measurable subset of T; if it is of measure 0 is is called *null*. An allocation $\mathbf{y}$ dominates an allocation $\mathbf{x}$ via a coalition S if $\mathbf{y}(t) \succ_t \mathbf{x}(t)$ for each $t \in S$ and S is *effective* for $\mathbf{y}$, i.e.

$$\int_S \mathbf{y} = \int_S \mathbf{i}.$$

The *core* is the set of all allocations that are not dominated via any nonnull coalition.

Competitive equilibrium

A *price vector* p is an n-tuple of nonnegative real numbers, not all of which vanish. A *competitive equilibrium* is a pair consisting of a price vector p and an allocation $\mathbf{x}$, such that for almost every trader t, $\mathbf{x}(t)$ is maximal with respect to $\succ_t$ in t's budget set.

Main theorem

Theorem

The core coincides with the set of equilibrium allocations.

Non-atomic measure

Definition

A measure μ is called non-atomic if for every measurable set A with $\mu(A) > 0$ there exists a measurable set B with $B \subset A$ and

$$0 < \mu(B) < \mu(A)$$

Theorem

The Lebesgue measure is nonatomic.

A consequence of non-atomicity

Theorem

If μ is a non-atomic measure, A is a measurable set with $\mu(A) > 0$, ϵ some positive number, then there exists a $B \subset A$ with

$$0 < \mu(B) < \epsilon$$

Theorem

If μ is a non-atomic measure, A is a measurable set with $\mu(A) > 0$, $0 < \epsilon < \mu(A)$ some positive number, then there exists a $B \subset A$ with

$$\mu(B) = \epsilon$$

Carathedory's Theorem

Definition

Given a set P its convex hull is the set of points x which can be written as

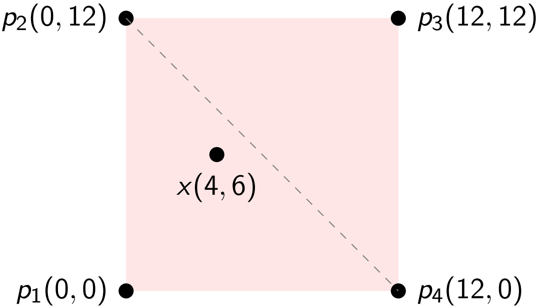
$$x = \lambda_1 y_1 + \cdots + \lambda_m y_m$$

for some $y_i \in P$, $\lambda_i \geq 0$ and $\sum \lambda_i = 1$.

Theorem

If a point $x \in \mathbb{R}^d$ lies in the convex hull of a set P, then x can be written as the convex combination of at most $d + 1$ points in P.

Caratheodory's Theorem (contd.)



$$x = (1/6)p_1 + (1/2)p_2 + (1/3)p_4$$

The proof

Let $\mathbf{x}$ be in the core. Define

$$\mathbf{F}(t) = \{y \colon y \succ_t \mathbf{x}(t)\}$$

$$\mathbf{G}(t) = \mathbf{F}(t) - \mathbf{i}(t)$$

For each set U of traders, let $\Delta(U)$ denote the convex hull of $\cup_{t \in U} \mathbf{G}(t)$. Define U to be full if its complement is null.

Lemma

There is a full set U of traders, such that 0 is not an interior point of $\Delta(U)$.

- N be the set of rational points r in $\mathbb{R}^n$ for which $\mathbf{G}^{-1}(r)$ is null. Define $U = T - \cup_{r \in N} \mathbf{G}^{-1}(r)$.
- If 0 is in the interior of $\Delta(U)$ then there is a $x > 0$ such that $-x \in \Delta(U)$ i.e. there exists $t_i, \ldots, t_k \in U$, positive $\beta_1, \ldots, \beta_k$ summing to 1, $x_i \in G_i^{-1}(t_k)$ such that

$$\sum_1^k \beta_i x_i = -x > 0.$$

Proof (contd.)

- By continuity we can find rational points r_i and positive rational numbers γ_i such that

$$r_i \in \mathbf{G}(t_i), \qquad \sum_{1}^{k} \gamma_i r_i = -r < 0$$

- Pick any trader t_0 is U and find a rational α large enough such that

$$\alpha r + \mathbf{i}(t_0) > \mathbf{x}(t_0)$$

- Set $r_0 = \alpha r$, $\alpha_0 = 1/(\alpha + 1)$, $\alpha_i = \alpha\gamma_i/(\alpha + 1)$. Then $\alpha_i > 0$, $\sum_0^k \alpha_i = 1$ and $\sum_0^k \alpha_i r_i = 0$, $r_i \in \mathbf{G}(t_i)$.

Proof (contd.)

- $\mathbf{G}^{-1}(r_i)$ is of positive measure for each i.
- For sufficiently small positive number δ we can find disjoint subsets S_i of $\mathbf{G}^{-1}(r_i)$ such that $\mu(S_i) = \delta\alpha_i$.
- Define coalition $S = \cup_0^k S_k$ and an assignment $\mathbf{y}$ by

$$\mathbf{y}(t) = \begin{cases} r_i + \mathbf{i}(t) & \text{t} \in S_i, \\ \mathbf{i}(t) & t \notin S \end{cases}$$