A Little Bit of Measure Theory

Lecture 4: Integration

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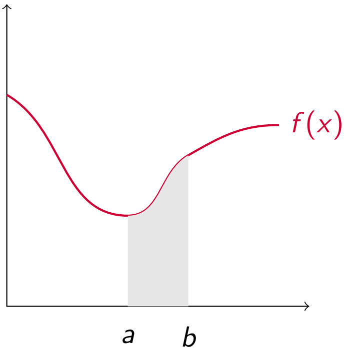
July 20, 2020

Integration



- The integral is the anti-derivative:

$$\int f(x)\,dx = F(x)$$
$$\Rightarrow F'(x) = f(x)$$

- The integral is the area under the graph of the function:

$$\int_a^b f(x)\,dx$$

The fundamental theorem of calculus

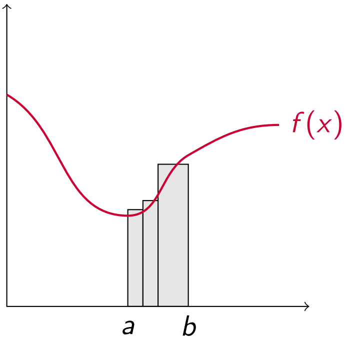
- $$\frac{d}{dx}\int_a^x f(u)\,du = f(x)$$
- $$\int_a^b f'(x)\,dx = f(b) - f(a)$$

The fundamental theorem of calculus

- $$\frac{d}{dx}\int_a^x f(u)\,du = f(x)$$
- $$\int_a^b f'(x)\,dx = f(b) - f(a)$$
- For which functions are the integrals defined?
- For which of them does the Fundamental Theorem(s) hold?
- Can we interchange limits and integrals?

$$\int \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \lim_{n\to\infty}\int f_n(x)\,dx$$

The Riemann integral



- Split the interval $[a, b]$ into subintervals.
- Approximate the areas by rectangles.
- See if there is a limit as the size of the subintervals shrink.

The Riemann integral (contd.)

- Everything works great for continuous functions with continuous derivatives.
- But this class is not large enough for applications.
- The class is not even stable under pointwise limits.

What use is the integral?

- Mathematical expectations: probability weighted sum.
  - Discrete case
$$\mathbb{E}[X] = \sum p_i x_i$$
  - Density case
$$E[X] = \int x f(x)\, dx$$
  - Where do these come from? What is a random variable and do all of them have a mathematical expectation?
- A sort of sum
$$U = \int_0^\infty e^{-\beta t} u(c(t))\, dt$$

What do we want from the integral?

We want a map $I\colon (M \to \mathbb{R}_+) \to \mathbb{R}_+$ (we will have to restrict the domain) which has the following properties:

- Consistency with the measure: For any measurable set A, $I(1_A) = \mu(A)$.
- Linearity: For any functions f and g and any scalars α and β

$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$$

- Monotonicity: If $f(x) \geq g(x)$ for all x, then $I(f) \geq I(g)$.
- Continuity: If $\lim_{n\to\infty} f_n(x) = f(x)$ for all x then

$$\lim_{n\to\infty} I(f_n) = I(f)$$

(we will have to put restrictions).

Simple functions

For some finite collection of disjoint measurable sets $A_1, \ldots, A_n$ and (extended) real numbers $c_1, \ldots, c_n$ let

$$s(x) = \begin{cases} c_i & x \in A_i \\ 0 & x \notin \cup_{i=1}^{n} A_i \end{cases}$$

Define

$$\int s(x)\, d\mu = \sum_{i=1}^{n} c_i \mu(A_i)$$

Assume $\infty \cdot 0 = 0$.

An example

- Consider $s\colon [0,1] \to \mathbb{R}$,

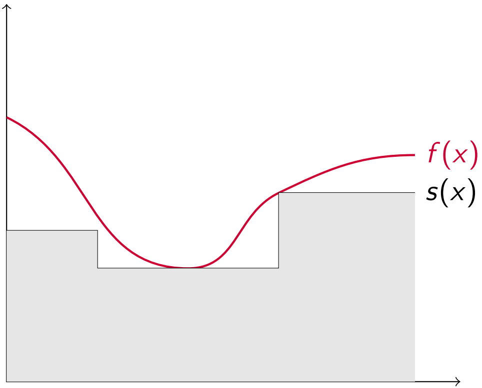
$$s(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

- Each point x is a measurable and has Lebesgue measure 0 since

$$\{x\} = \bigcap_{n=1}^{\infty} [x, x + 1/n)$$

- The set of rational numbers is a countable. Hence measure 0.
- Hence $\int s \, d\mu = 0$.
- s is not Riemann integrable.

Nonnegative functions



If f is a non-negative measurable function we define

$$\int f\, d\mu = \sup_s \int s\, d\mu$$

where the supremum is taken over all simple function s such that $s(x) \leq f(x)$ for all x.

Measurable functions

Definition

Given a measurable space $(M, \mathcal{M})$ a function $f\colon M \to \mathbb{R}_+$ is defined to be measurable if for any $\lambda \in \mathbb{R}$ we have

$$\{x\colon f(x) \leq \lambda\} \in \mathcal{M}$$

Equivalently

- $\{x\colon f(x) > \lambda\} \in \mathcal{M}$
- $\{x\colon f(x) < \lambda\} \in \mathcal{M}$ since

$$\{x\colon f(x) < \lambda\} = \bigcap_{n=1}^{\infty} \{x\colon f(x) \leq \lambda + 1/n\}$$

- $\{x\colon f(x) \geq \lambda\} \in \mathcal{M}$

There are many measurable functions

- If A is a measurable set then 1_A is measurable.
- If f and g are measurable functions and α is a scalar then αf, $f + g$ and fg, $\max(f, g)$, $\min(f, g)$ are measurable.
- Pointwise $\limsup$, $\liminf$ and $\lim$ of sequences of measurable functions are measurable.
- If ϕ is a continuous function and f is a measurable function then $\phi \circ f$ is measurable.
- All continuous functions are Lebesgue measurable.
- But all functions are not measurable in both the Lebesgue and infinite coin toss examples.

The integral works for nonnegative measurable functions

Restricted to non-negative measurable functions the integral has the following properties

- Consistency with the measure: For any measurable set A, $\int 1_A \, d\mu = \mu(A)$.
- Linearity: For any functions f and g and any scalars α and β

$$\int \alpha f \, d\mu + \int \beta g \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$$

- Monotonicity: If $f(x) \geq g(x)$ for all x, then $\int f \, d\mu \geq \int g \, d\mu$.
- Monotone convergence: If f_n is a sequence of non-negative measurable functions such that

$$f_1(x) \leq f_2(x) \leq \dots$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ for all x then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

.

An example of non-monotone convergence

- Suppose

$$f_n(\omega) = \begin{cases} 2^n & \text{if 1st to the } n\text{-th toss in } \omega \text{ are heads} \\ 0 & \text{otherwise} \end{cases}$$

- Then $\int f_n \, d\mu = 1$ for all n.
- $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ where

$$f(\omega) = \begin{cases} \infty & \text{if all the tosses in } \omega \text{ are heads} \\ 0 & \text{otherwise} \end{cases}$$

- $\int f \, d\mu = 0$.

Fatou's Lemma

Lemma

If f_n is a sequence of non-negative measurable functions then

$$\int (\liminf_{n\to\infty} f_n)\, d\mu \leq \liminf_{n\to\infty} \int f_n\, d\mu.$$

Definition

$$\liminf_{n\to\infty} a_n = \sup_n \left(\inf_{k\geq n} a_k \right)$$

Not necessarily non-negative functions

Let f be a real valued measurable function. Define,

$$f^+ = \max(f, 0), \qquad f^- = \max(-f, 0)$$

If $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$, define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

[The precondition can be written compactly as $\int |f| \, d\mu < \infty$. We then say that f is integrable.]

Dominated convergence

Theorem

Let f_n be a sequence of real-valued measurable functions such that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

If there is a non-negative function g such that

$$\int g \, d\mu < \infty, \qquad |f_n| < g \quad \textit{for all } n$$

then f is integrable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$