A Little Bit of Measure Theory

Lecture 3: The Strong Law of Large Numbers

Jyotirmoy Bhattacharya

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The (Strong) Law of Large Numbers

Theorem

If $X_i, i = 1, \ldots$ is a sequence of independent, identically distributed random variables with $\mathbb{E}[|X_i|] < \infty$ and $\mathbb{E}[X_i] = \mu$, then there is a set A with $\mathbb{P}(A) = 1$ such that for all $\omega \in A$

$$\lim_{n \to \infty} \frac{X_1(\omega) + \cdots + X_n(\omega)}{n} = \mu$$

Infinite coin tosses

Take

$$X_i(\omega) = \begin{cases} 1 & \text{If the } i\text{-th toss in } \omega \text{ is heads} \\ 0 & \text{If the } i\text{-th toss in } \omega \text{ is tails} \end{cases}$$

So what?

- Good: The limit is simple.
- Good: The limit is universal.
- Bad: In the long run we are all dead.

The "good" set

The subset

for every integer $k > 0$ there exists a N such that for all $n \geq N$ we have

$$|q_n(\omega) - 1/2| < 1/k$$

- For all $n \geq N$

$$\bigcap_{n=N}^{\infty} \{\omega\colon |q_n(\omega) - 1/2| < 1/k\}$$

- There exists some N, such that for all $n \geq N$

$$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega\colon |q_n(\omega) - 1/2| < 1/k\}$$

- For every k, there exists some N, such that for all $n \geq N$

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega\colon |q_n(\omega) - 1/2| < 1/k\} = G$$

Computing probabilities

$$G = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega \colon |q_n(\omega) - 1/2| < 1/k\}$$

$$G^c = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{\omega \colon |q_n(\omega) - 1/2| \geq 1/k\}$$

We will show that

$$\mathbb{P}(G^c) = 0.$$

The plan

$$G^c = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{\omega \colon |q_n(\omega) - 1/2| \geq 1/k\}$$

Let

$$A_{k,n} = \{\omega \colon |q_n(\omega) - 1/2| \geq 1/k\},$$

$$B_{k,N} = \bigcup_{n=N}^{\infty} A_{k,n},$$

and

$$C_k = \bigcap_{N=1}^{\infty} B_{k,N}.$$

Then

$$G^c = \bigcup_{k=1}^{\infty} C_k.$$

The plan (contd.)

$$A_{k,n} = \{\omega\colon |q_n(\omega) - 1/2| \geq 1/k\},$$

$$B_{k,N} = \bigcup_{n=N}^{\infty} A_{k,n}$$

$$C_k = \bigcap_{N=1}^{\infty} B_{k,N}$$

$$G^c = \bigcup_{k=1}^{\infty} C_k$$

- We see that $B_{k,N+1} \subset B_{k,N}$, so by using continuity from above

  $$\mathbb{P}(C_k) = \lim_{N\to\infty} \mathbb{P}(B_{k,N}).$$

  If we prove that the RHS is 0 we have $\mathbb{P}(C_k) = 0$.
- Then $\mathbb{P}(G^c) \leq \sum \mathbb{P}(C_k) = 0$ Since measures are nonnegative we get $\mathbb{P}(G^c) = 0$.

The plan (contd.)

$$A_{k,n} = \{\omega\colon |q_n(\omega) - 1/2| \geq 1/k\},$$

$$B_{k,N} = \bigcup_{n=N}^{\infty} A_{k,n}$$

- We have
$$\mathbb{P}(B_{k,N}) \leq \sum_{n=N}^{\infty} \mathbb{P}(A_{k,n})$$
- Find a sequence a_n such that
$$\mathbb{P}(A_{k,n}) \leq a_n$$
and $\sum_{n=1}^{\infty} a_n < \infty$.
- Then we can make use of
$$\lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = 0$$

Auxilliary functions

$$R_i(\omega) = \begin{cases} +1 & i\text{-th toss is heads in } \omega \\ -1 & i\text{-th toss is tails in } \omega \end{cases}$$

$$S_n(\omega) = \sum_{i=1}^{n} R_i(\omega)$$

$$\begin{aligned} S_n &= \text{No. of heads} - \text{No of tails} \\ &= \text{No. of heads} - (n - \text{No of heads}) \\ \text{No. of heads} &= (S_n + n)/2 \end{aligned}$$

$$q_n(\omega) = (\text{No. of heads})/n = S_n/2n + 1/2$$

Rewriting $A_{k,n}$

$$\begin{aligned} A_{k,n} &= \{\omega\colon |q_n(\omega) - 1/2| \geq 1/k\} \\ &= \{\omega\colon |S_n(\omega)| \geq 2n/k\} \end{aligned}$$

The Markov inequality

Theorem

If X is a nonnegative random variable then for any $\lambda > 0$

$$\mathbb{P}(X \geq \lambda) \leq \mathbb{E}[X]/\lambda$$

Proof.

Let A be the event $X \geq \lambda$.

$$\begin{aligned} X &= X \cdot (1_A + 1_{A^c}) \\ \mathbb{E}[X] &= \mathbb{E}[X \cdot 1_A] + \mathbb{E}[X \cdot 1_{A^c}] \\ &\geq \mathbb{E}[X \cdot 1_A] \\ &\geq \mathbb{E}[\lambda \cdot 1_A] = \lambda \mathbb{E}[1_A] \\ &= \lambda \mathbb{P}(A) \end{aligned}$$

Hence

$$P(A) \leq \mathbb{E}[X]/\lambda.$$

Applying Markov inequality to S_n^4

$$
\begin{aligned}
S_n^4 = (R_1 + \cdots + R_n)^4 = & R_1^4 + \cdots + R_n^4 \\
& + R_1^2 R_2^2 + \dots \\
& + R_1 R_2^3 + \dots \\
& + R_1 R_2 R_e^2 + \dots \\
& + R_1 R_2 R_3 R_4
\end{aligned}
$$

Expectations of powers of R_i

$$R_i(\omega) = \begin{cases} +1 & i\text{-th choice is heads in } \omega \\ -1 & i\text{-th choice is tails in } \omega \end{cases}$$

$$\mathbb{E}[R_i] = 0, \mathbb{E}[R_i^2] = 1, \mathbb{E}[R_i^3] = 0, \mathbb{E}[R_i^4] = 1$$

Applying Markov inequality to S_n^4 (contd.)

$$
\begin{aligned}
\mathbb{E}[S_n^4] = \mathbb{E}[R_1^4] + \cdots + \mathbb{E}[R_n^4] \\
+ \mathbb{E}[R_1^2 R_2^2] + \ldots \\
+ \mathbb{E}[R_1 R_2^3] + \ldots \\
+ \mathbb{E}[R_1 R_2 R_e^2] + \ldots \\
+ \mathbb{E}[R_1 R_2 R_3 R_4]
\end{aligned}
$$

Applying Markov inequality to S_n^4 (contd.)

Using independence,

$$\begin{aligned}\mathbb{E}[S_n^4] &= \mathbb{E}[R_1^4] + \cdots + \mathbb{E}[R_n^4] \\ &+ \mathbb{E}[R_1^2]\,\mathbb{E}[R_2^2] + \ldots \\ &+ \mathbb{E}[R_1]\,\mathbb{E}[R_2^3] + \ldots \\ &+ \mathbb{E}[R_1]\,\mathbb{E}[R_2]\,\mathbb{E}[R_e^2] + \ldots \\ &+ \mathbb{E}[R_1]\,\mathbb{E}[R_2]\,\mathbb{E}[R_3]\,\mathbb{E}[R_4]\end{aligned}$$

using the previous results

$$\begin{aligned}&= \mathbb{E}[R_1^4] + \cdots + \mathbb{E}[R_n^4] \\ &+ \mathbb{E}[R_1^2]\,\mathbb{E}[R_2^2] + \ldots \text{3n(n-1) terms} \\ &= n + 3n(n-1) = 3n^2 - 2n \leq 3n^2\end{aligned}$$

Counting the terms

- We are looking at ways of forming terms of type $R_iR_jR_kR_l$ with two indices repeated twice each.
- The index for the first position can be chosen in n ways.
- It can be repeated in one of the remaining 3 places.
- The remaining two places have to be filled by a common index different from the first one. This can be chosen in $n-1$ ways.

Applying Markov inequality to S_n^4 (contd.)

$$\begin{aligned}
\mathbb{P}(|S_n| \geq 2n/k) &= \mathbb{P}(|S_n|^4 \geq (2n/k)^4) \\
&\leq \frac{\mathbb{E}[S_n^4]}{(2n/k)^4} \\
&\leq \frac{3n^2}{(2n/k)^4} \\
&= \frac{3k^4}{16n^2}
\end{aligned}$$

So if we take $a_n = \frac{3k^4}{16n^2}$ then $\mathbb{P}(A_{k,n}) \leq a_n$ and $\sum_{n=0}^{\infty} a_n < \infty$. We are done.

$1/n^2$ converges

$$
\begin{aligned}
&1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \dots \\
&\le 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{8^2} + \dots \\
&= 1 + \frac{2}{2^2} + \frac{4}{4^2} + \dots \\
&= 1 + \frac{1}{2} + \frac{1}{4} + \dots \\
&= 2
\end{aligned}
$$

Weak law of large numbers

Definition (Convergence in probability)

A sequence of random variables X_n converges in probability to a random variable Y if for any $\epsilon > 0$ it is the case that

$$\lim_{n\to\infty} \mathbb{P}(|X_n - Y| > \epsilon) = 0$$

Theorem

If $X_i, i = 1, \dots$ is a sequence of independent, identically distributed random variables with $\mathbb{E}[|X_i|] < \infty$ and $\mathbb{E}[X_i] = \mu$, then

$$\frac{X_1(\omega) + \cdots + X_n(\omega)}{n}$$

converges in probability to μ.

From Strong to Weak

Define

$$A_n = \{|S_n/n| > \epsilon\}$$

We have from the strong law

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

Use continuity from above.