A Little Bit of Measure Theory

Lecture 2

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The Problem

- We have a probability assignment for finitely determined sets of the sort $(T?H??\ldots)$.
- We would like to calculate probabilities for more complicated sets like

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega \colon |q_n(\omega) - 1/2| < 1/k\} = G$$

Caratheodory's Theorem

Definition

Given a set X, a set $\mathcal{A}$ of subsets of X is called a semiring if

1. $\emptyset \in \mathcal{A}$.
2. $A, B \in \mathcal{A}$ implies $A \cap B \in \mathcal{A}$.
3. If $A, B \in \mathcal{A}$ and $A \subset B$ then there exists disjoint $\mathcal{A}$-sets $C_1, \ldots, C_n$ such that $B - A = \cup_{k=1}^{n} C_k$

Theorem (Billingsley, 11.3)

Suppose that μ_0 is a set function on a semiring $\mathcal{A}$. Suppose that μ_0 has values in $[0, \infty]$ and that $\mu_0(\emptyset) = 0$, and that μ_0 is finitely additive and countably subadditive. Then μ_0 extends to a measure μ on a σ-algebra $\mathcal{M}$ containing $\mathcal{A}$.

Additivity and subadditivity

Definition

A set function μ_0 defined on a family of sets $\mathcal{A}$ is finitely additive if whenever $A_i, i = 1, \ldots, N$ are *disjoint* sets in $\mathcal{A}$ such that $\cup_{i=1}^n A_i \in \mathcal{A}$ we have

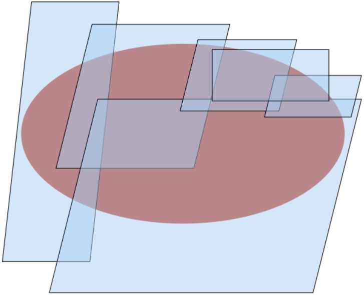
$$\mu_0\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu_0(A_i)$$

Definition

A set function μ_0 defined on a family of sets $\mathcal{A}$ is countably subadditive if whenever $A_i, i = 1, \ldots$ is a sequence of sets in $\mathcal{A}$ such that $\cup_{i=1}^\infty A_i \in \mathcal{A}$ we have

$$\mu_0\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \mu_0(A_i)$$

Proof: Outer measure



Definition

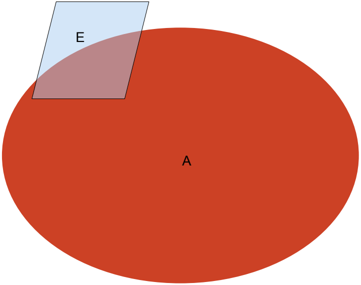
Given a set function μ_0 defined on some collection $\mathcal{A}$ of subsets of some sets M we define the outer measure $\mu^*(E)$ for any subset E of M by

$$\mu^*(E) = \inf \sum_n \mu_0(A_i)$$

where the infimum is over all finite and countable collections A_i of A sets such that

$$E \subset \bigcup_n A_i$$

Proof: Caratheodory's criteria



Definition

A subset A of M is measurable if for any subset E of M we have

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E)$$

Proof: the rest

- The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains $\mathcal{A}$.
- μ^* is finitely additive on $\mathcal{M}$.
- μ^* equals μ_0 on $\mathcal{A}$
- Declare $\mu = \mu^*$ on $\mathcal{M}$

Finitely determined sets are a semiring

The collection of finitely determined sets plus the null set is a semiring.

1. The null set is included by definition.
2. $$(T?H??\ldots) \cap (T?TT?\ldots) = \emptyset$$
$$(T?H??\ldots) \cap (TH???\ldots) = (THH??\ldots)$$
3. $$\begin{aligned}(T???\ldots) - (THT?\ldots) &= (TTT?\ldots)\\ &\cup (TTH?\ldots)\\ &\cup (THH?\ldots)\end{aligned}$$

Finite additivity

Let μ be the set function that has value 0 on $\emptyset$ and the value 2^{-n} on a finitely determined set which fixes n positions. Then it is finitely additive.

$$\begin{aligned}(T???\ldots) &= (TH??\ldots) \cup (TTH?\ldots) \cup (TTT?\ldots)\\ &= (THT?\ldots)\\ &\cup (THH?\ldots)\\ &\cup (TTH?\ldots)\\ &\cup (TTT?\ldots)\end{aligned}$$

Also finite additivity implies finite subadditivity.

Compactness

Theorem

Let E be a finitely determined set and let $\mathcal{G}$ be a collection of finitely determined sets. If

$$E \subset \bigcup_{G \in \mathcal{G}} G$$

then there exists a finite collection of set $G_1, \ldots, G_n$ from $\mathcal{G}$ such that

$$E \subset \bigcup_{i=1}^{n} G_i$$

"Every cover has a finite subcover."

Proof

1. Suppose $E = (T??\ldots)$ is covered by a collection $\mathcal{G}$ of finitely determined sets but there is no finite cover of E in $\mathcal{G}$.
2. Then either $(TH?\ldots)$ or $(TT?\ldots)$ does not have a finite subcover (or both). Suppose $(TT?\ldots)$ does not have a finite subcover.
3. Then either $(TTH?\ldots)$ or $(TTT?\ldots)$ does not have a finite subcover, and so on.

Proof (contd)

1. Going on like this forever we have a sequence of symbols $a_2, a_3, \ldots$ such that
   1.1 $(Ta_2??\ldots)$ does not have a finite subcover.
   1.2 $(Ta_2a_3??\ldots)$ does not have a finite subcover.
   1.3 And so on..
2. Now consider $\omega = (Ta_2a_3a_4\ldots)$. This surely belongs to $E = (T??\ldots)$.
3. Since $\mathcal{G}$ covers E, there must be a $G \in \mathcal{G}$ such that $\omega \in G$.
4. Since G is finitely determined then it must be of the form $(Ta_2?a_4\ldots a_k??\ldots)$ where k is the last position determined.
5. But then $(Ta_2\ldots a_k??\ldots)$ has a finite subcover, namely $\{G\}$.

Countable subadditivity

Suppose

$$A = \bigcup_{i=1}^{\infty} A_i$$

Then by compactness there is a finite number N such that

$$A = \bigcup_{i=1}^{N} A_i$$

By finite subadditivity

$$\mu(A) \leq \sum_{i=1}^{N} \mu(A_i)$$

By non-negativity

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

The other big example

1. Half open intervals on the line $[a, b)$.
2. Semiring:
   2.1 $[1,3) \cap [2,5) = [2,3)$.
   2.2 $[1,5) - [2,3) = [1,2) \cup [3,5)$
3. Set function: $\mu([a, b)) = b - a$.
4. Countable subadditivity uses topology of the line.
5. Result: Lebesgue measure.

Lebesgue measurable sets

- Open intervals:

$$(a, b) = \cup_{n=1}^{\infty}[a + 1/n, b)$$

- Open sets: every open set is a countable union of open intervals.
- Closed sets: complements of open sets.
- The Borel σ-algebra: the smallest σ-algebra containing all open (eqiv. closed) sets.

Nonmeasurable sets

- For a finite set of positions I let $\tau_I(\omega)$ be the sequence obtained by flipping the outcomes at I.
- Declare $\omega_1 \sim \omega_2$ if there is a finite I such that $\omega_1 = \tau_I(\omega_2)$. This is a equivalence relation:
  - $\omega = \tau_\emptyset(\omega)$
  - $\omega_1 = \tau_I(\omega_2)$ implies $\tau_I(\omega_1) = \omega_2$
  - $\omega_1 = \tau_I(\omega_2)$ and $\omega_2 = \tau_J(\omega_3)$ implies $\omega_1 = \tau_{I \Delta J}(\omega_3)$
- Let V be a set formed by taking exactly one element from each equivalence class. Is it measurable?

Nonmeasurable set (contd)

- If V is measurable then because of unbiasedness and independence $\mu(\tau_I(V)) = \mu(V)$ for any I.
- We have
$$\Omega = \bigcup_I \tau_I(V)$$
- This is a countable disjoint union (check!), so we must have
$$\mu(\Omega) = \sum_I \mu(\tau_I(V)) = \sum_I \mu(V)$$
- Contradiction!