A Little Bit of Measure Theory

Lecture 1

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A coin tossing example

True or False

For an infinite sequence of tosses of an independent unbiased coin, the proportion of heads always tends to $1/2$.

A coin tossing example

True or False

For an infinite sequence of tosses of an independent unbiased coin, the proportion of heads always tends to $1/2$.

False

- $HHH\dots$
- $HTTHTT\dots$

A coin tossing example

True or False

For an infinite sequence of tosses of an independent unbiased coin, the proportion of heads always tends to $1/2$.

False

- $HHH\ldots$
- $HTTHTT\ldots$

Modification

For an infinite sequence of tosses of an independent unbiased coin, the proportion of heads, with high probability, tends to $1/2$.

Try bottom-up probability assignment

- Because the coin is unbiased and the tosses are independent, each sequence of outcomes is as likely as any other sequence.
- What probability do we attach to each sequence:
  - 0: Problem.
  - > 0: Problem.

Top-down probability assignment

Finitely determined subsets

- Let $T????\ldots$ denote the subset of sequences where the first toss comes out tails and the rest of the tosses can be anything.
- Let $?T?H???\ldots$ denote the subset of sequences were the second toss is tails and the fourth toss is heads and the rest of the tosses can be anything.
- Subsets like these where the outcome at a finite number of positions is specified and the rest are left free we will call finitely-determined subsets.

Top-down probability assignment

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Assigning probabilities

By unbiasedness and independence a finitely-determined subset which specifies the outcomes at n positions will have probability 2^{-n}.

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Problem

Many subsets are not finitely determined.

Limit of the proportion of heads

Definition

For an infinite sequence of tosses ω, define

$$q_n(\omega) = \frac{\text{No. of heads in the first } n \text{ positions in } \omega}{n}.$$

Problem

Is the set of "good" sequences

$$G = \{\omega\colon \lim_{n\to\infty} q_n(\omega) = 1/2\}$$

finitely determined?

Approximating

$$\lim_{n\to\infty} q_n(\omega) = 1/2$$

if and only if for every $\epsilon > 0$ there exists a N such that for all $n \geq N$ we have

$$|q_n(\omega) - 1/2| < \epsilon$$

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In fact it is enough that for every integer $k > 0$ there exists a N such that for all $n \geq N$ we have

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Thus,

$$G = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega\colon |q_n(\omega) - 1/2| < 1/k\}$$

Approximating (contd.)

The subset

for every integer $k > 0$ there exists a N such that for all $n \geq N$ we have

$$|q_n(\omega) - 1/2| < 1/k$$

Approximating (contd.)

The subset

for every integer $k > 0$ there exists a N such that for all $n \geq N$ we have

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- For all $n \geq N$

$$\bigcap_{n=N}^{\infty} \{\omega \colon |q_n(\omega) - 1/2| < 1/k\}$$

Approximating (contd.)

The subset

for every integer $k > 0$ there exists a N such that for all $n \geq N$ we have

$$|q_n(\omega) - 1/2| < 1/k$$

- For all $n \geq N$

$$\bigcap_{n=N}^{\infty} \{\omega : |q_n(\omega) - 1/2| < 1/k\}$$

- There exists some N, such that for all $n \geq N$

$$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |q_n(\omega) - 1/2| < 1/k\}$$

Approximating (contd.)

The subset

for every integer $k > 0$ there exists a N such that for all $n \geq N$ we have

$$|q_n(\omega) - 1/2| < 1/k$$

- For all $n \geq N$

$$\bigcap_{n=N}^{\infty} \{\omega\colon |q_n(\omega) - 1/2| < 1/k\}$$

- There exists some N, such that for all $n \geq N$

$$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega\colon |q_n(\omega) - 1/2| < 1/k\}$$

- For every k, there exists some N, such that for all $n \geq N$

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega\colon |q_n(\omega) - 1/2| < 1/k\} = G$$

The task

- Each of the sets

$$\{\omega\colon |q_n(\omega) - 1/2| < 1/k\}$$

  are finitely determined.
- But to go from them to G we have to take infinite unions and intersections.
- Two problems: possibility of extending the probability assignment and calculating probabilities.

σ-algebra

Definition

Given a set X, a collection of subsets $\mathcal{X}$ of X is called a σ-algebra on X if

- $X \in \mathcal{X}$.
- If $A \in \mathcal{X}$ then $A^c \in \mathcal{X}$.
- If $A_i \in \mathcal{X}$ for $i = 1, 2, \ldots$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{X}$.

Note

Intersection of sequences belong by De Morgan's laws.

Example

- $\mathcal{X} = \{X, \emptyset\}$
- $\mathcal{X}$ = Powerset of X.

Measure

Definition

Given a set M and a σ-algebra $\mathcal{M}$ on M, a measure μ on $(M, \mathcal{M})$ is a function from elements of $\mathcal{M}$ to the set of extended real numbers such that:

- $\mu(A) \geq 0$, for all $A \in \mathcal{M}$.
- $\mu(\emptyset) = 0$.
- If A_n, $n = 1, \ldots,$ is a sequence of disjoint sets in $\mathcal{M}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Examples

Length, area, volume. Probability measures:$\mu(M) = 1$.

Some consequences

- Monotonicity. If $E, F \in \mathcal{M}$ and $E \subset F$ then $\mu(E) \leq \mu(F)$.
- Subadditivity. If $E_n \in \mathcal{M}$, $n = 1, \ldots$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{1}^{\infty} \mu(E_n).$$

- Continuity from below. If $E_n \in \mathcal{M}$, $n = 1, \ldots$ and $E_1 \subset E_2 \subset \ldots$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

- Continuity from above. If $E_n \in \mathcal{M}$, $n = 1, \ldots,$ and $E_1 \supset E_2 \supset \ldots$ and $\mu(E_1) < \infty$ then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

A Paradox?

- Any individual sequence of outcomes is an intersection of a decreasing sequence of finitely determined sets:

$$\begin{aligned} TTHT\ldots &= T?????\ldots \\ &\cap TT????\ldots \\ &\cap TTH???\ldots \\ &\ldots \end{aligned}$$

- So each individual sequence belongs to any σ-algebra containing the finitely determined sets.
- Each must have probability 0.
- How does probability add up to 1?

Countable & Uncountable

Definition

A set is called countable if it can be put in a one-to-one correspondence with the set of natural numbers.
A set that is neither countable nor finite is called uncountable.

The set of outcomes is uncountable

Suppose we have a sequence

1. TFTT...
2. TFTF...
3. FTFT...
4. ...

Not in the sequence

FTT...

Surprise!

- Rationals are countable

$$
\begin{aligned}
0/1 \\
0/2, 1/1 \\
0/3, 1/2, 2/1 \\
0/4, 1/3, 2/2, 3/1 \\
\ldots
\end{aligned}
$$

- Reals in $[0, 1]$ are not

$$
\begin{aligned}
&0.1324\ldots \\
&0.5749\ldots \\
&\ldots
\end{aligned}
$$

Goals of the theory

- Construct measures: start from measures on a class of sets whose measures are given by some outside data. Extend consistently to larger classes of sets.
- Integrate: take real valued functions on a set which has a measure defined on it. Assign a number to each such function in a way which captures the notion of 'weighted sum' with the weights provided by the measure.

Uses of the theory

- Foundations of probability: Remove the need to treat discrete and continuous random variables separately, also deal with variables which are neither. Give a general meaning to conditioning. Limit theorems.
- A good theory of integration: Integrate more functions. Behave nicely under limits.